The Colebrook-White Formula for Friction Factors in the Transition Region

The Colebrook-White (CW) formula [1] is often used to determine the Darcy (or Moody) friction factor, \( f \), for pipes. The formula is shown below:

\[
\frac{1}{\sqrt{f}} = -2 \cdot \log \left( \frac{k}{3.7} + \frac{2.51}{N_Re \cdot \sqrt{f}} \right)
\]

where

\[ k = \frac{\varepsilon}{ID} \]

relative roughness

\[ \varepsilon \]

absolute roughness

\[ ID \]

internal pipe diameter

\[ N_Re \]

Reynolds number

The formula is implicit in \( f \) requiring an iterative method to determine \( f \). To simplify computations, numerous alternative formulas have been proposed that compute \( f \) explicitly. These formulas have been checked for accuracy against the CW formula by various investigators. Recently, Fred Lusk [2] very thoroughly evaluated 25 formulas. As with many of the previous surveys, the objective was to determine how well each formula compared with the CW formula, not with actual data. I became curious about the accuracy of the CW formula itself.

The CW formula was developed by combining two other formulas, the Prandtl-von Karman formula for smooth pipes, and the von Karman-Nikuradse formula for fully turbulent flow in rough pipes. You will usually see these formulas in a different form involving a simple, non log term. That term has been converted to the forms below.

Prandtl-von Karman

\[
\frac{1}{\sqrt{f}} = -2 \cdot \log \left( \frac{2.51}{N_Re \cdot \sqrt{f}} \right)
\]

(2)

von Karman-Nikuradse

\[
\frac{1}{\sqrt{f}} = -2 \cdot \log \left( \frac{k}{3.7} \right)
\]

(3)

The Prandtl-von Karman formula has a theoretical basis, but the numbers in the formula are based on data. The von Karman-Nikuradse formula was proposed by von Karman and then later the parameters were obtained based on the data of Nikuradse [3]. By summing the arguments of the log terms, Colebrook and White obtained a formula that satisfies the
Prandtl-von Karman formula when $k = 0$, and that approaches the von Karman-Nikuradse formula when $N_{Re}$ becomes very large (i.e. for turbulent flow). This leaves the following question: how well does the CW formula perform at intermediate values of $N_{Re}$ when $k$ is not zero? This region is called the transition region. It is not to be confused with the critical region between laminar flow and the beginning of turbulent flow, $2100 < N_{Re} < 4000$.

I have not yet obtained the Colebrook paper [1] containing the source of the CW formula, but I have read an earlier paper by Colebrook and White [4] that addressed the differences between the behavior of commercial pipes and the sand roughened pipes used by Nikuradse. The Nikuradse data showed the friction factor curves approaching the fully turbulent plateaus from below as $N_{Re}$ is increased. In contrast, data by others using galvanized and wrought iron pipes show the friction factor falling to the plateau as $N_{Re}$ is increased. Colebrook and White conducted experiments with various artificial roughness patterns. They concluded that the commercial pipe behaviour may be due to non-uniform roughness.

The commercial pipe data shown by Colebrook and White [4] appeared to justify the CW formula that came later. However, the data were in graphical form and were not suitable for numerical comparisons with the CW formula. One of the studies cited in [4] was performed by Freeman [5] in 1892. Freeman conducted extensive tests using water and a number of pipes. The data are available in tables and graphs.

In the rough pipe category, Freeman studied 13 wrought iron pipes with diameters between 1/4 in. and 8 in. Two cast iron pipes, 4 and 8 in. were also studied. I used those data to check the accuracy of the CW formula.

**Methodology**

1. Relative roughness values, $k$, were calculated via CW for each pipe using the friction factor data at the highest $N_{Re}$ for that pipe. This avoided using generic absolute roughnesses for wrought and cast iron.
2. The friction factors at $N_{Re} = 4000$ (lowest value for CW allowed) and 10,000 were computed via CW. This step tests the amount of change in $f$ for each pipe between the low and high $N_{Re}$ values. It also checks CW at an intermediate flow rate.
Wrought iron pipes

A 5 in. and an 8 in. pipe were also studied, but the results were identical to the 4 in. and 6 in. pipes, respectively.

Schedule 40 wrought steel pipe assumed in Cameron Hydraulic Data, same as carbon steel in Crane. This assumption does not affect the computation of \( k \) or \( f \).

A Mathcad solve block is used to compute the value of \( k \). The algorithm is solved in a parametric manner with \( N \) and \( f \) as parameters.
\[ k_{CW} := k \left( N_{\text{high}} \cdot f_{\text{high}} \right) \quad \text{relative roughness for each pipe} \]

\[
\begin{bmatrix}
0.006 \\
0.005 \\
0.002 \\
0.002 \\
0.001 \\
0.003 \\
9.048 \cdot 10^{-4} \\
4.586 \cdot 10^{-4} \\
4.988 \cdot 10^{-4} \\
4.558 \cdot 10^{-4} \\
4.558 \cdot 10^{-4}
\end{bmatrix}
\]

\[ k_{CW} = \begin{bmatrix}
1.671 \cdot 10^{-4} \\
2.566 \cdot 10^{-4} \\
1.452 \cdot 10^{-4} \\
1.584 \cdot 10^{-4} \\
1.386 \cdot 10^{-4} \\
3.387 \cdot 10^{-4} \\
1.558 \cdot 10^{-4} \\
9.435 \cdot 10^{-5} \\
1.275 \cdot 10^{-4} \\
1.529 \cdot 10^{-4} \\
2.304 \cdot 10^{-4}
\end{bmatrix} \text{ft} \quad \epsilon_{\text{ave}} := \text{mean} (\epsilon) = (1.787 \cdot 10^{-4}) \text{ ft}
\]

Although not needed for the validation, the absolute roughness using assumed IDs is determined below.

The roughness of carbon steel is 0.00015 ft per Crane [6], but no value for wrought iron is provided.
Using the k values determined at the high flow rates, compute f at low flow rates and compare with the data.

\[
f := .01
\]

\[
f = \frac{1}{\left(-2 \cdot \log \left( \frac{k}{3.7} + \frac{2.51}{N \cdot \sqrt{f}} \right) \right)^2}
\]

\[
f(k,N) := \text{find}(f)
\]

define percent error function

\[
err(x_{\text{data}}, x_{CW}) := \frac{x_{CW} - x_{\text{data}}}{x_{\text{data}}} \cdot 100
\]

wrought iron at N\_Re = 4000

\[
f_{4000\_CW} := f\left(k_{CW\_p}, 4000\right)
\]

<table>
<thead>
<tr>
<th>prediction</th>
<th>data</th>
<th>percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.045</td>
<td>0.046</td>
<td>-0.718</td>
</tr>
<tr>
<td>0.045</td>
<td>0.045</td>
<td>-1.4</td>
</tr>
<tr>
<td>0.042</td>
<td>0.044</td>
<td>-4.541</td>
</tr>
<tr>
<td>0.042</td>
<td>0.043</td>
<td>-2.093</td>
</tr>
<tr>
<td>0.041</td>
<td>0.042</td>
<td>-2.575</td>
</tr>
<tr>
<td>0.042</td>
<td>0.042</td>
<td>-3.28</td>
</tr>
<tr>
<td>0.041</td>
<td>0.042</td>
<td>-4.336</td>
</tr>
<tr>
<td>0.04</td>
<td>0.042</td>
<td>-4.241</td>
</tr>
<tr>
<td>0.04</td>
<td>0.042</td>
<td>-4.343</td>
</tr>
</tbody>
</table>

\[
f_{4000\_CW} = [0.042, 0.042, 0.041, 0.042, 0.04, 0.042, 0.04, 0.042, 0.04, 0.042]
\]

\[
f_{4000} = [0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042]
\]

\[
err(f_{4000}, f_{4000\_CW}) = [-0.465, -3.28, -4.336, -4.241, -4.343, -4.343]
\]
wrought iron at $N_{Re} = 10,000$

$$f_{10k_{CW}} = f\left(k_{CW_p}, 10000\right)$$

<table>
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</thead>
<tbody>
<tr>
<td>[0.038]</td>
<td>[0.039]</td>
<td>[-2.716]</td>
</tr>
<tr>
<td>0.038</td>
<td>0.038</td>
<td>-1.905</td>
</tr>
<tr>
<td>0.034</td>
<td>0.037</td>
<td>-7.993</td>
</tr>
<tr>
<td>0.034</td>
<td>0.035</td>
<td>-5.001</td>
</tr>
<tr>
<td>0.033</td>
<td>0.033</td>
<td>-2.161</td>
</tr>
<tr>
<td>0.035</td>
<td>0.035</td>
<td>-1.978</td>
</tr>
<tr>
<td>0.032</td>
<td>0.033</td>
<td>-3.464</td>
</tr>
<tr>
<td>0.032</td>
<td>0.033</td>
<td>-5.443</td>
</tr>
<tr>
<td>0.032</td>
<td>0.033</td>
<td>-5.262</td>
</tr>
<tr>
<td>0.032</td>
<td>0.033</td>
<td>-5.455</td>
</tr>
</tbody>
</table>

Cast iron pipes

$$D_{ci} := \left[\frac{4}{8}\right] \cdot \text{in} \quad ID_{ci} := \left[\frac{4.026}{7.981}\right] \cdot \text{in}$$

$$f_{4000} := \left[.0422\right] \quad f_{10000} := \left[.0334\right] \quad f_{high} := \left[.0241\right] \quad N_{high} := \left[600000\right]$$

$$p := 0 . . 1$$

$$k_{ci_{CW}} := k\left(N_{high}, f_{high}\right) = \left[0.002 \quad 2.302 \cdot 10^{-4}\right]$$

$$\varepsilon_{ci} := k_{ci\_CW} \cdot ID_{ci} = \left[7.132 \cdot 10^{-4} \quad 1.531 \cdot 10^{-4}\right] \text{ft}$$

The absolute roughness value in Crane for cast iron is 0.0008 ft.

cast iron at $N_{Re} = 4000$

$$f_{ci\_4000_{CW}} := f\left(k_{ci\_CW_p}, 4000\right)$$

<table>
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<tr>
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<th>percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.042]</td>
<td>[0.042]</td>
<td>[-0.444]</td>
</tr>
<tr>
<td>0.04</td>
<td>0.042</td>
<td>-4.881</td>
</tr>
</tbody>
</table>
cast iron at N_Re = 10,000

\[ f_{ci_{10k,CW}} = f(k_{ci,CW}, 10000) \]

<table>
<thead>
<tr>
<th></th>
<th>prediction</th>
<th>data</th>
<th>percent error</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ f_{ci_{10k,CW}} ]</td>
<td>0.034</td>
<td>0.033</td>
<td>1.694</td>
</tr>
<tr>
<td>[ f_{10000} ]</td>
<td>0.033</td>
<td>0.033</td>
<td>-6.478</td>
</tr>
</tbody>
</table>

**Conclusions:**

1. The friction factors in commercial pipe decline from the value at N_Re = 4000 to the fully turbulent value as depicted by the CW formula.
2. The CW formula provided good (< 10% error) estimates of the friction factors in the transition region for the commercial pipes.
3. It appears that the variability of the unknown absolute roughness of a commercial pipe might impart significant error in the prediction of \( f \).

**References:**


